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Large deformation and mechanics of flexible isotropic membrane ballooning in three dimensions by differential quadrature method †

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Abstract

This paper presents a computationally efficient and accurate new methodology in the differential quadrature analysis of structural mechanics for flexible membranes ballooning in three dimensions under a negative air pressure differential. The differential quadrature method is employed to discretize the resulting equations in the axial direction as well as for the solution procedure. The weighting coefficients employed are not exclusive, and any accurate and efficient method such as the generalized differential quadrature method may be used to produce the methods weighting coefficients. A second-order paraboloid of revolution is assumed to describe the ballooning shape. This study asserts the accuracy, convergency, and efficiency of the methodology by solving some typical stability, straining analysis membrane problems, and comparing the results with those of the exact solutions and/or those of physical tests.

***Keywords*:** Ballooning; Differential quadrature method; Isotropic; Large deformation; Membrane

1. Introduction

Membrane structures have long been under consideration as structural elements because they possess certain advantageous characteristics such as high strength-to-weight ratio and relatively low cost. However, what makes their design difficult is the inherent non-linearity of the deformation and the material. A mechanically fastened single-ply membrane on a roofing system is a common example. The theory of mechanics and performance of the tensile structure has been well established by Otto [1] and many others.

We use ideal models of perfectly flexible membrane; it is common to have a flexible membrane laid on a stiff substrate and mechanically attached. Fastener pullout is a performance concern, and this has been studied by Baskaran [2] and Gerhardt [3]. Zarghamee [4, 5] studied the wind force acting on the ballooning roof membrane and concluded that the membrane must be designed for the static as well as for a part of the fluctuating component of wind pressure. Much research work has been published on the development of an appropriate wind testing protocol for the single-ply membrane [6-8]. Several authors have investigated membrane in the screen-type wall system. However, most of the works are restricted and related to studies on the ballooning of the single-ply roof membrane and its effect on roofing system performance. Ballooning seems to be an unpopular issue, and it has not been the subject of much research. Burnett [9] discussed seven important functions that the exterior membrane performs in the screen-type wall system. Kumar [10] performed a comprehensive literature review on pressure-equalized rain screen walls. In the published literature on pressure equalization models, some do not include the air cavity volume as an independent parameter, therefore excluding the influence of the membrane ballooning [11-15]. Choi [16-18] carried out a series of full-scale tests to study aluminum-curtain wall systems often used in

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Southeast Asia. Canadian Mortgage and Housing Corporation (CMHC) [17, 18] developed a computer program, Rain Screen 2.1, to assess screen pressure equalization. The effect of membrane ballooning on pressure equalization has been considered in some pressure equalization models [15-18]. Furthermore, the importance of the flexibility of the membrane and its ballooning in the screen-type wall system has been realized and discussed in some published literature [9, 10].

The commonly used numerical methods in structural mechanics analysis are the finite element method, finite difference method, boundary element method, and Rayleigh–Ritz method. As an efficient alternative numerical tool, Differential Quadrature Methods (DQMs) have been used for structural analysis. Details on their development and applications are found in the review paper by Bert and Malik [19]. Among the different approaches for determining the weighting coefficients [19], the Lagrange interpolation functions and trigonometric/harmonic functions are the most popular test functions. The applications of PDQ for thin beams and plates as well as for rectangular thick plates have been carried out by Malekzadeh [20, 21]. Liew and Han [22] employed the DQ method to present the bending analysis of simply supported thick skew plates based on the first-order shear deformation plate theory.

In this paper, the capabilities of the DQMs for the deformation analysis of flexible membranes ballooning in three dimensions under a negative air pressure differential are investigated. There is clearly a need to study the correlation between the wind pressure acting on the membrane and its ballooning, particularly the ballooning shape and the deflection. This paper applies the theory of elastics to membrane ballooning and discretizes the resulting equations in the axial direction by DQM, and for the solution procedure, develops a theory to model the ballooning under a given static air pressure differential.

2. Governing differential equations

In accordance with the theory of elasticity, a valid solution can be obtained if force equilibrium (stress analysis) is satisfied. Consider a *3D* ballooning membrane element in the *xyz*-coordinate system as shown in Fig. 1. Under the negative air pressure differential P, the membrane deforms to form a curvature, and the membrane element is formed by intersecting pairs of

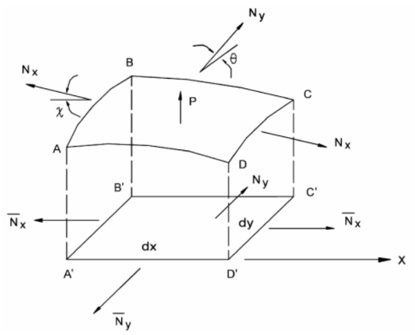

Fig. 1. Ballooning membrane element for stress analysis.

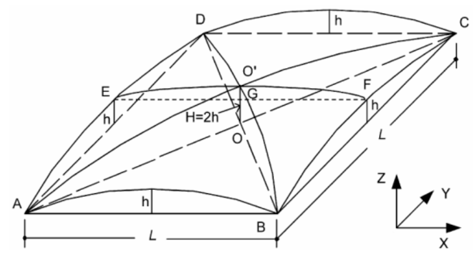

Fig. 2. A flexible membrane ballooning in a 3-D situation.

coordinate lines x = const and y = const. We describe the membrane stresses by a system of skew forces Nx and Ny (throughout this formulation, the notation in [25] is used).

The fundamental mechanical characteristics of the 3-D flexible membrane need to be considered before developing the structural mechanics. Several assumptions need to be made. First, in order to apply the theory of elasticity, the 3-D ballooning membrane is assumed to deform within its linear-elastic range. In Fig. 2, the membrane's four fasteners are idealized as four fixed points, and the air pressure differential P is applied across the membrane. When ballooning, plane *ABCD* deforms to a 3-D curvature in the *xyz*-coordinate system, as shown in the figure. The air pressure differential P is perpendicular to the deformed membrane curvature, that is, along the normal direction of the curvature. Given that the depth of the wall cavity limits the membrane deformation, the ratio of the span, that is, the ratio of the distance between adjacent mechanical fasteners to the maximum deflection of the membrane in a real screen-type wall system, is usually greater than 20. It follows that the decomposed force in the z-direction is much greater than that in the *x*- and *y*-directions. It is presumed that considering only the air pressure differential component in the *z*-direction will not cause any significant error.

Fig. 2 illustrates a geometric model of the 3-D ballooning membrane. It shows four edges (*AB, BC, CD*, and *DA*) and two diagonals (*AC* and *BD*) as straight lines before the membrane balloons. Under the negative air pressure differential *P*, the membrane balloons to form a curvature. It is assumed that the four edges and two diagonals all become second-order parabolas after the membrane deforms. The maximum deflection of the membrane is the distance between points *O* and *O',* which is denoted by *H*. The maximum deflection for four-edge parabolas is referred to as *h*. If a straight line is drawn through points *E* and *F*, it intersects with line *OO'*. Parabola *EO'F* and parabola BFC have the same span; thus, the deflection of parabola *EO'F* is also *h*.

Therefore, the following geometric relation shown in Eq. (1) between *H* and *h* is true in

$$H = 2h . \tag{1}$$

In fact, if one uses a plane perpendicular to the *x–y* plane to cut the membrane; the cutting trace is always a second-order parabola. In analytic geometry, this kind of surface is referred to as the *paraboloid of revolution*. The ballooning membrane shown in Fig. 2 is cut from a paraboloid of revolution by four planes: $x=\pm (l/2)$ and $y=\pm (l/2)$.

Its shape function is given by

$$x^2 + y^2 = \frac{L^2}{2H}(L - z) . \tag{2}$$

The skew forces *Nx* and *Ny* are the forces per unit length of the line elements through which they are transmitted. The actual forces are obtained by multiplying them by the length of this element. The element length *dx* and *dy* become *dx/cos w* and *dy/cos y* after the membrane deforms. Therefore, we can write the horizontal components of these forces, that is, the *x* component, as follows:

$$N_X \frac{dy}{\cos\theta}\cos\chi = \overline{N_X}dy \tag{3}$$

The *y* component is given by

$$N_Y \frac{dx}{\cos\chi}\cos\theta = \overline{N_Y}dx \tag{4}$$

The relationship between the membrane forces and their projections on the *x–y* plane is described by

$$N_x = \frac{\cos\theta}{\cos\chi}\overline{N_x} \; and \; N_y = \frac{\cos\chi}{\cos\theta}\overline{N_y} \tag{5}$$

As the geometric shape of the ballooning membrane is already determined in Eq. (2), we can write it as follows:

$$tg\theta = \frac{\partial z}{\partial y} = \frac{-4H}{L^2}y \; and \; tg\chi = \frac{\partial z}{\partial x} = \frac{-4H}{L^2}x \tag{6}$$

According to the relationship between triangular functions, we obtain

$$\begin{aligned} \cos\theta &= \frac{1}{\sqrt{1+(tg\theta)^2}} = \frac{1}{\sqrt{1+(16H^2/L^4)^2 y^2}} \\ \cos\chi &= \frac{1}{\sqrt{1+(tg\chi)^2}} = \frac{1}{\sqrt{1+(16H^2/L^4)^2 x^2}} \end{aligned} \tag{7}$$

Substituting Eq. (7) into Eq. (5) gives

$$\begin{aligned} N_x &= \frac{\sqrt{1+(16H^2/L^4)^2 x^2}}{\sqrt{1+(16H^2/L^4)^2 y^2}}\overline{N_x} \\ N_y &= \frac{\sqrt{1+(16H^2/L^4)^2 y^2}}{\sqrt{1+(16H^2/L^4)^2 x^2}}\overline{N_y} \end{aligned} \tag{8}$$

After these preparations, it becomes easy to write the conditions of the equilibrium for the membrane element shown in Fig. 1. Considering only the external pressure differential *P*, we only need to write the equilibrium equation in the *z*-direction. The *z*-direction components of the forces acting on the membrane element are given by

$$\begin{aligned} &(N_x \frac{dy}{\cos\theta})\sin\chi(dx) = \overline{N_x}(tg\chi)dxdy = \overline{N_x}\frac{\partial z}{\partial x}dxdy \\ &(N_y \frac{dx}{\cos\chi})\sin\theta(dy) = \overline{N_y}(tg\theta)dxdy = \overline{N_y}\frac{\partial z}{\partial y}dxdy \end{aligned} \tag{9}$$

The differential equation for force equilibrium involves their differential increments shown as follows:

$$\frac{\partial}{\partial x}(\overline{N_x}\frac{\partial z}{\partial x})dxdy + \frac{\partial}{\partial y}(\overline{N_y}\frac{\partial z}{\partial y})dxdy + Pdxdy = 0 \tag{10}$$

Calculate to yield

$$\overline{N_x}\frac{\partial^2 z}{\partial x^2}+\overline{N_y}\frac{\partial^2 z}{\partial y^2}+\frac{\partial \overline{N_x}}{\partial x}\frac{\partial z}{\partial x}+\frac{\partial \overline{N_y}}{\partial y}\frac{\partial z}{\partial y}+P=0 \quad (11)$$

As there is no shear force or external load in the *x*- or *y*-directions, the equilibriums in the *x*- and *y*-directions are indicated as

$$\frac{\partial \overline{N_x}}{\partial x}=0 \; and \; \frac{\partial \overline{N_y}}{\partial y}=0 \quad (12)$$

Therefore, Eq. (11) can be simplified as

$$\overline{N_x}\frac{\partial^2 z}{\partial x^2}+\overline{N_y}\frac{\partial^2 z}{\partial y^2}+P=0 \quad (13)$$

3. Review of the differential quadrature method

The DQM is based on the idea that the partial derivative of a field variable at the ith discrete point in the computational domain is approximated by a weighted linear sum of the values of the field variable along the line that passes through that point, which is parallel with the coordinate direction of the derivative [23]. Considering a two-dimensional field variable *u (x,y)*, its mth order derivative with respect to *x*, and its *(m+n)*th order derivative with respect to *x* and y are approximated as follows:

$$\left.\frac{\partial^m u}{\partial x^m}\right|_{(x_i,y_j)}=\sum_{k=1}^{N_x}\overline{A}_{ik}^{(m)}u(x_k,y_j) \quad (14)$$

$$\left.\frac{\partial^{m+n} u}{\partial x^m \partial y^n}\right|_{(x_i,y_j)}=\sum_{k=1}^{N_x}\overline{A}_{ik}^{(m)}\overline{A}_{jl}^{(n)}u(x_k,y_l) \quad (15)$$

There are two key points in the successful application of the DQM: how the weighting coefficients are determined and how the grid points are selected. The method developed by Shu and Richard [23] is said to be computationally more accurate than other methods. According to Shu and Richard's rule, the weighting coefficients of the first-order derivatives in ξ-direction (ξ = *x* or *y*) are determined as follows:

$$\overline{A}_{ij}^{(1)}=\begin{cases}\dfrac{M(\xi_i)}{(\xi_i-\xi_j)M(\xi_j)} & for \;\; i\neq j \\ -\sum_{j=1,i\neq j}^{N_\xi}\overline{A}_{ij}^{(1)} & for \;\; i=j\end{cases}, \quad (16)$$

where

$$M(\xi_i)=\prod_{j=1,i\neq j}^{N_\xi}(\xi_i-\xi_j) \quad (17)$$

It has been demonstrated that non-uniform grid points give better results with the same number of equally spaced grid points [19]. In this paper, we chose these sets of grid points in terms of the natural coordinate directions *x* and *y* as

$$\xi_i=\frac{1}{2}\left[1-\cos\left[\frac{(i-1)\pi}{(N_\xi-1)}\right]\right]. \quad (18)$$

4. Differential quadrature analogs

The derivatives of the field variables may be transformed into the computational domain efficiently by modifying the weighting coefficients as

$$A_{ij}^x=\frac{\overline{A_{ij}^x}}{L}, B_{ij}^x=\frac{\overline{B_{ij}^x}}{L^2}, A_{ij}^y=\frac{\overline{A_{ij}^y}}{L}, B_{ij}^y=\frac{\overline{B_{ij}^y}}{L^2} \quad (19)$$

Using the general differential quadrature (GDQ) of Shu and Richards using (13) and (19), the DQ analogs of the governing equations and boundary conditions take the following forms, respectively.

4.1 Governing equations

Substituting Eq. (14) and (15) into Eq. (13) and rearranging them, we can obtain

$$(\overline{N_x})_{ij}\sum_{k=1}^{N_x}B_{ik}^x Z_{kj}+(\overline{N_y})_{ij}\sum_{k=1}^{N_y}B_{jk}^y Z_{ik}+P_{ij}=0 \quad (20)$$

where

$$(\;)_{ij}=(\;)\Big|_{(x_i,y_j)},\;(\;)_{ij}^{(\xi)}=\frac{\partial(\;)}{\partial \xi}\Big|_{(x_i,y_j)}. \quad (21)$$

4.2 Boundary conditions

Let us consider the stress resultants *Nx* and *Ny* in the diagonal parabola (*AC* in Fig. 2). Any point on line *AC* must satisfy *x* = *y* on the *x–y* plane; thus, it follows that Eq. (23) is true.

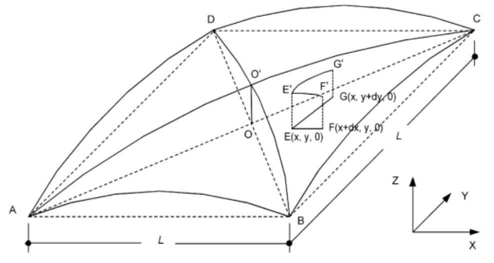

Fig. 3. Line elements on the ballooning membrane for strain analysis.

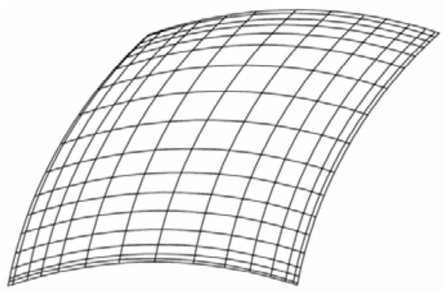

Fig. 4. Deformed membrane.

$$(\overline{N_x})_{ij} = (\overline{N_y})_{ij} \tag{22}$$

Note that Eq. (23) is only true for the diagonal parabola AC. The shear strain is not considered here because the membrane is shear-free. The deformed line elements $E'F'$ and $E'G'$ can be considered as straight lines, which are sufficient approximations for the strain analysis purpose. The coordinates of points E', F', and G' are as follows:

$$E'(x, y, z(x, y)), F'(x+dx, y, z(x+dx, y)) \tag{23}$$

$$G'(x, y+dy, z(x, y+dy)) \tag{24}$$

The deformed length of the line element $E'F'$ is (Fig. 3)

$$|E'F'| = \sqrt{(dx)^2 + (\frac{\partial z}{\partial x})^2 (dx)^2} = \sqrt{1 + (\frac{\partial z}{\partial x})^2}\, dx\ . \tag{25}$$

The length of the line element EF before ballooning is simply dx. Therefore, the strain is described as

$$(\varepsilon_x)_{ij} = \frac{|E'F'| - |EF|}{|EF|} = \sqrt{1 + (\sum_{k=1}^{N_x} A_{ik}^x Z_{kj})^2} - 1 \tag{26}$$

$$(\varepsilon_y)_{ij} = \frac{|E'G'| - |EG|}{|EG|} = \sqrt{1 + (\sum_{k=1}^{N_y} A_{jk}^y Z_{ik})^2} - 1 \tag{27}$$

Table 1. Material properties.

Material property	Tyvek™ Homewrap	Typar™ Housewrap
Poisson's ratio	0.24	0.29
Thickness t (mm)	.016	0.33
E value (MPa)	875	250

Table 2. The maximum deflection H (mm) of the 3-D ballooning membrane.

P (Pa)	Maximum deflection H (mm)							
	Tyvek™ Homewrap (E = 875 MPa, t = 0.16mm)				Typar™ Housewrap (E = 250 MPa, t = 0.33mm)			
	L = 406mm (16″ grid)		L = 610mm (24″ grid)		L = 406mm (16″ grid)		L = 610mm (24″ grid)	
	Analytical	DQM	Analytical	DQM	Analytical	DQM	Analytical	DQM
100	14.1	13.97	23.4	23.21	16.4	16.26	27.4	27.15
200	17.3	17.16	28.6	28.34	20.5	20.35	32.0	31.71
300	19.7	19.91	31.3	31.6	23.4	23.58	39.3	39.57
400	21.7	21.92	32.9	33.06	25.6	25.75	42.3	41.96
500	23.3	23.11	39.1	39.41	27.3	27.10	44.5	44.81
600	24.7	24.95	41.0	40.67	28.6	28.37	50.8	50.39

Hooke's law has the standard form shown as

$$Et\varepsilon_x = N_x - \nu N_y, Et\varepsilon_y = N_y - \nu N_x, Et\gamma_{xy} = 2(1+\nu)N_{xy}\ . \tag{28}$$

Solving the above equations, a large deformation can be obtained. In the DQM, the governing equations and boundary conditions are directly discretized; thus, elements of stiffness and mass matrices are evaluated directly. In addition to satisfying the governing equations in each domain, the external boundary conditions as well as geometric and natural compatibility conditions at common sections of the neighboring sub-domains or elements are enforced.

5. Numerical examples

In this section, two commonly used types of membranes in the US market, namely, Tyvek™ Homewrap and Typar™ Housewrap are solved by DQM. Without any loss of generality, an equal number of grid points in both directions are assumed, that is, $Nx = Ny$.

Chamara [23] conducted experimental work to determine the material properties of these two membranes, and they are the parameters needed to solve Eq. (22). The results are summarized in Table 1. For analytical purposes, the fastener is idealized as a point. To check the convergence and accuracy of the algorithm of DQM for membranes, an example is considered. Its results are compared with the solutions of Xing Shi [24], in which he analytically calculated the maximum deflections. In accordance with theory of elasticity, Xing Shi substituted the following: force equilibrium (Stress Analysis), deformation compatibility, and stress–strain relationship, triggering a valid analytical solution. Using the mathematical software to solve the mentioned equations for Tyvek^{TM} Homewrap and Typar^{TM} Housewrap, the results of the maximum deflection H under different air pressure differentials are summarized in Table 2. It is shown that by using 10 grid points, the error is less than 1%. Furthermore, the stability of the convergence behaviors of the solutions is confirmed.

6. Conclusions

A numerical methodology called DQ procedure for the deformation of flexible isotropic membrane was developed. The completeness of the present DQM was demonstrated for a membrane under a negative air pressure differential. Accurate results were obtained with only few grid points, showing the advantage of the low computation cost of the method (Fig. 4.).

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